International Journal of Theoretical Physics, Vol. 31, No. 6, 1992

Finsler Gauge Transformations and General Relativity

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Received October 22, 1991

The theory of gauge transformations in Finsler space is applied to general relativity. It is seen that the transformations produce new metrics which correspond to the introduction of physical fields. The geodesic equation in the transformed space is equivalent to the equation of motion in the original space where the field is included by a force term. An example is given of a transformation and resulting metric in which the electromagnetic potential is related to parameters of the gauge transformation rather than to gauge potentials. This implies that the electromagnetic field corresponds to a connection instead of a curvature. Another example is given which shows how Weyl or conformal transformations are related to a class of the gauge transformations.

1. INTRODUCTION

In recent years a theory of gauge transformations in the context of Finsler space has been developed by G. S. Asanov and collaborators (Asanov, 1985, 1987, 1988; Aringazin and Asanov, 1988; Asanov *et al.*, 1988; Asanov and Kiselev, 1988) and also by S. Ikeda (Ikeda, 1985, 1987, 1989). In this theory the Finsler tangent vectors are treated as independent variables attached to points in space-time. The homogeneous transformations of the tangent space are called gauge transformations. Standard fiber bundle or gauge theory methods can be applied to the Finsler theory to produce, for example, identification of Finsler connections with generalized gauge potentials.

The theory is significant because it offers an alternative to multidimensional theories of the Kaluza-Klein type, yet is able to achieve the same sorts of objectives, for example, a unified approach to Yang-Mills fields and the space of general relativity. The theory is broad enough to include practically all fields of current physical interest into its framework (Asanov and Kiselev, 1988).

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0020-7748/92/0600-1025$06.50/0

The present paper proposes a specialized application of the Finsler gauge theory to general relativity itself. The fiber bundle includes the general relativistic base space with local coordinates x^μ and a fiber space $\Pi^{-1}(x)$ which has local coordinates y^μ, the Finsler tangent vectors. In the general Asanov theory the metrics in both the base space and the tangent space are considered. Here, attention is directed only to the base space metric, which is, however, dependent on both x^μ and y^μ. The action on this metric of tangent space transformations is the main focus of this work. It will be seen that this approach gives a new point of view on the spaces of general relativity and provides a framework for the direct incorporation of fields (for example, the electromagnetic field) into the metric.

The procedure to be followed is to define a transformation and consider its effect on the geometric objects in the space. In general, as is usual in gauge theories, expressions will be sought for derivatives which are covariant under the gauge transformations. This leads to the connections or gauge potentials.

In Section 2 coordinate transformations in Finsler space are reviewed. The approach of Miron and Anastasiei (1987) is applied. This gives a concise derivation of the Cartan connection which reflects the horizontal and vertical subspace decomposition of the Finsler tangent bundle. This is accomplished by the introduction of a nonlinear connection which defines a covariant basis for the space.

In Section 3 the development of the main ideas of this work is presented. The gauge transformations now act only on the tangent vectors and not on the base space coordinates. These transformations are able to generate new metrics in the spaces of general relativity by a process which might be thought of as "turning on" the gauge fields. A new nonsymmetric connection appears which is directly related to the gauge transformations. A new velocity is also defined which is related to the original velocity by a scale change rather than a contravariant transformation.

In Section 4 an equivalence principle is used to define a special class of gauge transformations in the tangent space where the original metrics are locally Lorentzian. These transformations have particular significance in that they describe how forces in an inertial frame can be represented by a metric. The nonsymmetric connection of the metric can be expressed directly in terms of the tangent space transformation matrices. The geodesic equation in the transformed space with a connection related to the gauge potentials can be the same as the equation of motion in the original inertial space which includes the external force.

In Section 5 particular examples of transformations are chosen in order to illustrate the theory. The first transformation example shows how the gauge transformation parameters can be related to the electromagnetic potential. This means that the electromagnetic potential is interpreted as a different sort of geometrical object than the gauge potential. Consequently, the electromagnetic field is similar geometrically to a connection instead of to a curvature as in previous theories. The equation of motion is then identical with the Lorentz equation for charged particles. In other theories the equation of geodesic deviation is compared with the Lorentz equation. So a new unified approach to gravitation and electromagnetism is advanced.

The second transformation example has the form of a conformal transformation. This compares with traditional Weyl theory except that the crucial objection, of noninvariance of scalar products, is removed. So the conformal transformation considered as a Finsler gauge transformation offers a way to reinstate the Weyl theory without nonphysical effects.

In regard to notation, the reader will quickly observe that the presentation here uses the older tensor index formalism where all quantities are written explicitly in terms of local coordinates. This is not as elegant as modern geometrical notation, but has the dual advantages of operational facility and of being accessible to a broader range of physicists.

2. TRANSFORMATIONS IN FINSLER SPACE

The standard references in the theory of Finsler space are the books of Rund (1959) and Asanov (1985). Here, the general approach of Miron and Anastasiei (1987) which emphasizes the fiber bundle aspects of Finsler space is used. Chapter VII of Miron and Anastasiei (1987) contains a rigorous geometrical definition of Finsler spaces. Since this monograph is not widely accessible, the discussion will be repeated in some detail.

The Finsler metric function F is defined by $F^2(x, y) = g_{\mu\nu}(x, y)y^\mu y^\nu$. The tensor $g_{\mu\nu}$ is not in general the metric tensor, but simply a homogeneous tensor of degree zero in y which is used for the purpose of defining F. The Finsler metric tensor is $f_{\mu\nu}(x, y) = \frac{1}{2}\partial^2 F^2/\partial y^\mu\, \partial y^\nu$.

The relation between f and g is

$$f_{\mu\nu} = g_{\mu\nu} + \frac{\partial g_{\mu\alpha}}{\partial y^\nu} y^\alpha + \frac{\partial g_{\alpha\nu}}{\partial y^\mu} y^\alpha + \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial y^\mu\, \partial y^\nu} y^\alpha y^\beta \tag{2.1}$$

It is not hard to see that if g is independent of y, then $f = g$ and the space is Riemannian.

The function F is homogeneous of degree one with respect to y. By the Euler theorem this means that $(\partial F/\partial y^\alpha)y^\alpha = F$. This implies

$$2C_{\alpha\beta\mu} y^\mu = \frac{\partial f_{\alpha\beta}}{\partial y^\mu} y^\mu = 0 = \frac{\partial g_{\alpha\beta}}{\partial y^\mu} y^\mu \tag{2.2}$$

The quantity $C_{\alpha\beta\mu}=\frac{1}{2}(\partial f_{\alpha\beta}/\partial y^{\mu})$ is known as the Cartan torsion tensor and is obviously symmetric in all its indices.

It is natural to define the Finsler line element as $ds=(f_{\mu\nu}\,dx^{\mu}\,dx^{\nu})^{1/2}$. Here dx is no longer a general increment in coordinate space, but is in a direction tangent to the path defined by s. The direction is understood to be timelike in the present work.

As indicated in the Introduction, the metric tensor here determines distances in the coordinate space only. In the more general work of Asanov (1987) and Asanov *et al.* (1988), metrics of both the coordinate space and the tangent space are considered. The tangent space or "internal" metric is not discussed here.

Since F is homogeneous and $y^{\mu}=dx^{\mu}/ds$,

$$ds=F(x,dx)=F(x,y)\,ds \tag{2.3}$$

Variation of F produces the Euler-Lagrange equations,

$$\frac{d}{ds}\left[\frac{\partial F(x,y)}{\partial y^{\mu}}\right]-\frac{\partial F(x,y)}{\partial x^{\mu}}=0$$

This, in turn, as shown in any treatise on Finsler space, leads to the Finsler geodesic equation:

$$\frac{dy^{\mu}}{ds}+\gamma^{\mu}_{\alpha\beta}y^{\alpha}y^{\beta}-\frac{1}{F}\frac{dF}{ds}y^{\mu}=0$$

The Finsler-Christoffel connection $\gamma(x,y)$ is

$$\gamma^{\mu}_{\alpha\beta}=\frac{1}{2}f^{\mu\nu}\left(\frac{\partial f_{\alpha\nu}}{\partial x^{\beta}}+\frac{\partial f_{\nu\beta}}{\partial x^{\alpha}}-\frac{\partial f_{\alpha\beta}}{\partial x^{\nu}}\right)$$

Since from (2.3) $F(x,y)$ has the value of unity, the geodesic equation in this case is

$$\frac{dy^{\mu}}{ds}+\gamma^{\mu}_{\alpha\beta}y^{\alpha}y^{\beta}=0 \tag{2.4}$$

A change of section or coordinate transformation of the base space is $x'^{\mu}=x'^{\mu}(x^{\nu})$. In local coordinates the transformation matrix is denoted by $X^{*\mu}_{\nu}=\partial x'^{\mu}/\partial x^{\nu}$.

Since $dx'^{\mu}=X^{*\mu}_{\nu}\,dx^{\nu}$, the transformation of y is $y'^{\mu}=X^{*\mu}_{\nu}y^{\nu}$.

The natural basis $(\partial/\partial x^{\mu},\ \partial/\partial y^{\mu})$ of the module of the vector fields transforms as

$$\frac{\partial}{\partial x'^{\mu}}=X^{\nu}_{\mu}\frac{\partial}{\partial x^{\nu}}+\frac{\partial X^{\nu}_{\mu}}{\partial x'^{\alpha}}y'^{\alpha}\frac{\partial}{\partial y^{\nu}} \qquad \frac{\partial}{\partial y'^{\mu}}=X^{\nu}_{\mu}\frac{\partial}{\partial y^{\nu}}$$

where $X^{\mu}_{\nu}X^{*\nu}_{\alpha}=\delta^{\mu}_{\alpha}$.

A new basis is needed which transforms covariantly. This is $(\delta/\delta x^\mu, \partial/\partial y^\mu)$, with

$$\frac{\delta}{\delta x^\mu}=\frac{\partial}{\partial x^\mu}-N^\nu_\mu\frac{\partial}{\partial y^\nu}$$

The matrix N^ν_μ is the local representation of what is called the nonlinear connection:

$$N^\mu_\nu=\frac{1}{2}\frac{\partial}{\partial y^\nu}(\gamma^\mu_{\alpha\beta}y^\alpha y^\beta)$$

The N^ν_μ satisfy the transformation law

$$N'^\alpha_\beta=X^{*\alpha}_\nu X^\mu_\beta N^\nu_\mu+X^{*\alpha}_\nu\frac{\partial X^\nu_\beta}{\partial x'^\mu}y'^\mu$$

so the tangent basis transforms as required.

The dual basis is given by $(dx^\mu, \delta y^\mu)$ with $\delta y^\mu = dy^\mu + N^\mu_\nu\, dx^\nu$.

The geodesics of the Finsler space can be expressed as

$$\frac{\delta y^\mu}{\delta s}=\frac{dy^\mu}{ds}+N^\mu_\alpha\frac{dx^\alpha}{ds}=0,\qquad y^\mu=\frac{dx^\mu}{ds} \tag{2.5}$$

The partial derivative of a general vector $q^\mu(x, y)$ would transform as

$$\frac{\partial q'^\mu}{\partial x^\alpha}=X^{*\mu}_\nu\frac{\partial q^\nu}{\partial x^\alpha}+\frac{\partial X^{*\mu}_\nu}{\partial x^\alpha}q^\nu$$

This should be replaced by a derivative

$$q'^\mu_{;\alpha}=X^{*\mu}_\nu X^\beta_\alpha q^\nu_{;\beta}\qquad q^\nu_{;\beta}=\frac{\delta q^\nu}{\delta x^\beta}+F^\nu_{\alpha\beta}q^\alpha \tag{2.6}$$

where F is a connection which transforms as

$$F'^\mu_{\alpha\beta}=X^{*\mu}_\nu X^\gamma_\alpha X^\delta_\beta F^\nu_{\gamma\delta}-X^\gamma_\alpha X^\delta_\beta\frac{\partial X^{*\mu}_\delta}{\partial x^\gamma}$$

Of particular interest is the covariant derivative of the Finsler metric tensor,

$$f_{\alpha\beta;\mu}=\frac{\delta f_{\alpha\beta}}{\delta x^\mu}-f_{\nu\beta}F^\nu_{\alpha\mu}-f_{\alpha\nu}F^\nu_{\beta\mu} \tag{2.7}$$

where

$$\frac{\delta f_{\alpha\beta}}{\delta x^\mu}=\frac{\partial f_{\alpha\beta}}{\partial x^\mu}-N^\nu_\mu\frac{\partial f_{\alpha\beta}}{\partial y^\nu}$$

It is usually assumed that the space is "metric," that is, $f_{\alpha\beta;\mu}=0$, so that permutation of the indices in (2.7) gives

$$F^{\nu}_{\alpha\beta}=\frac{1}{2}f^{\mu\nu}\left(\frac{\delta f_{\mu\alpha}}{\delta x^{\beta}}+\frac{\delta f_{\beta\mu}}{\delta x^{\alpha}}-\frac{\delta f_{\alpha\beta}}{\delta x^{\mu}}\right) \tag{2.8}$$

This means that F is symmetric in the lower two indices. Also, it follows immediately from (2.2) and (2.8) that

$$F^{\nu}_{\alpha\beta}y^{\beta}=\gamma^{\nu}_{\alpha\beta}y^{\beta}-N^{\mu}_{\beta}y^{\beta}C^{\nu}_{\alpha\mu}$$

The use of (2.2) again gives

$$F^{\nu}_{\alpha\beta}y^{\beta}y^{\alpha}=\gamma^{\nu}_{\alpha\beta}y^{\beta}y^{\alpha}$$

Also, when y is substituted for q in (2.6), since y is independent of x, one has

$$N^{\nu}_{\alpha}=y^{\beta}F^{\nu}_{\alpha\beta} \tag{2.9}$$

It can now be seen that the two expressions of the geodesic equation (2.4) and (2.5) are equivalent.

A vertical covariant derivative can also be defined:

$$q^{\mu}|_{\alpha}=\frac{\partial q^{\mu}}{\partial y^{\alpha}}+q^{\beta}C^{\mu}_{\alpha\beta}$$

It follows from the definition of C that

$$f_{\alpha\beta}|_{\mu}=\frac{\partial f_{\alpha\beta}}{\partial y^{\mu}}-f_{\nu\beta}C^{\nu}_{\alpha\mu}-f_{\alpha\nu}C^{\nu}_{\mu\beta}=0$$

The triad $(N^{\alpha}_{\beta}, F^{\nu}_{\alpha\beta}, C^{\nu}_{\alpha\beta})$ is called the Cartan connection of the Finsler space. The connection is naturally expressed in terms of horizontal and vertical parts. The connections are obviously interrelated, however, as expressed in the equations immediately above.

A number of different fields, torsions, and curvatures can be defined from these connections. One of the references given at the beginning of this section should be consulted for these developments.

Once the Cartan connection is determined for the fiber bundle, one can then consider a section of the fiber for particular coordinates x^{μ}. When this is set, y^{μ} may no longer be an independent variable, but can be considered to be a vector field $y^{\mu}(x)$. Thus, $y^{\mu}(x)$ plays the role of an auxiliary vector field. This defines an immersed submanifold in a sense recently described by Miron and Kawaguchi (1991).

The Finsler metric $f_{\mu\nu}(x, y)$ in this case gives rise to a tensor field which is a Riemannian metric tensor $a_{\mu\nu}(x)=f_{\mu\nu}(x, y(x))$. The space with metric

$a_{\mu\nu}$ is called the osculating Riemann space related to the Finsler space (Asanov, 1985, pp. 32–33 and 110ff.).

Since

$$\frac{\partial a_{\mu\nu}}{\partial x^\alpha}=\frac{\partial f_{\mu\nu}}{\partial x^\alpha}+\frac{\partial f_{\mu\nu}}{\partial y^\beta}\frac{\partial y^\beta}{\partial x^\alpha}$$

the Christoffel connection for the Riemann space is

$$\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}=\gamma^\mu_{\alpha\beta}+\frac{\partial y^\nu}{\partial x^\alpha}C^\mu_{\beta\nu}+\frac{\partial y^\nu}{\partial x^\beta}C^\mu_{\alpha\nu}-a^{\mu\gamma}\frac{\partial y^\nu}{\partial x^\gamma}C_{\alpha\beta\nu}$$

The geodesic equation for the osculating Riemann space is obviously then

$$\frac{dy^\mu}{ds}+\left\{\begin{matrix}\mu\\ \alpha\beta\end{matrix}\right\}y^\alpha y^\beta=0$$

This compares with (2.4).

3. TANGENT SPACE TRANSFORMATIONS

Attention is now directed to a class of gauge transformations which act on the tangent space. These are local changes of coordinates in the fiber itself. The nature of the transformation group is initially left unspecified. Particular examples will be considered later.

The tangent vector y^μ transforms as

$$\bar{y}^\mu=Y^{*\mu}_\nu y^\nu \tag{3.1}$$

which is specialized here to $Y^{*\mu}_\nu=\partial\bar{y}^\mu/\partial y^\nu$.

Even though the transformation does not act on the base space coordinates, it will be seen to produce changes in this space. Also, the transformation itself is x-dependent, i.e., $Y^{*\mu}_\nu=Y^{*\mu}_\nu(x,y)$.

A similar type of transformation has been studied by Ikeda [see, in particular, Ikeda (1985), but also Ikeda (1987, 1989)]. Note, however, that the unification scheme proposed by Ikeda is not applied here.

It is assumed here that the partial derivative operator transforms as a covariant vector,

$$\frac{\partial}{\partial\bar{y}^\mu}=Y^\nu_\mu\frac{\partial}{\partial y^\nu} \tag{3.2}$$

$$Y^{*\mu}_\alpha Y^\alpha_\nu=\delta^\mu_\nu \tag{3.3}$$

In general, (3.2) and (3.3) would not hold together because the brackets $[\partial/\partial\bar{y}^\mu, \partial/\partial\bar{y}^\nu]$ would not vanish. Here, however, the metric condition (3.7) will be obtained, so that

$$\frac{\partial\bar{y}^\mu}{\partial\bar{y}^\nu} = Y_\nu^\beta \frac{\partial}{\partial y^\beta}(Y_\alpha^{*\mu} y^\alpha) = Y_\nu^\beta Y_\alpha^{*\mu}\delta_\beta^\alpha + Y_\nu^\beta \frac{\partial Y_\alpha^{*\mu}}{\partial y^\beta} y^\alpha = \delta_\nu^\mu$$

The second term vanishes due to the metric condition. More directly, $Y_\nu^{*\mu} = \partial\bar{y}^\mu/\partial y^\nu$ and (3.1) imply that $(\partial Y_\alpha^{*\mu}/\partial y^\beta)y^\alpha = 0$.

Also, due to the definition (3.1), the condition $\partial Y_\nu^{*\mu}/\partial y^\alpha = \partial Y_\alpha^{*\mu}/\partial y^\nu$ must hold. This, too, relates to the fact that the basis $\partial/\partial\bar{y}^\mu$ is holonomic.

It is assumed that the tensor used to form the Finsler metric function transforms as

$$\bar{g}_{\mu\nu}(x, y) = Y_\mu^\alpha Y_\nu^\beta g_{\alpha\beta}(x, y) \tag{3.4}$$

This implies that the Finsler metric function itself is scalar under the transformation:

$$\bar{F}^2(x, \bar{y}) = \bar{g}_{\mu\nu}\bar{y}^\mu\bar{y}^\nu = g_{\alpha\beta} Y_\mu^\alpha Y_\nu^\beta Y_\gamma^{*\mu} y^\gamma Y_\sigma^{*\nu} y^\sigma = g_{\alpha\beta} y^\alpha y^\beta = F^2(x, y)$$

The covariant vector associated with y is $y_\mu = \frac{1}{2}(\partial F^2/\partial y^\mu)$, which transforms as

$$\bar{y}_\mu = Y_\mu^\alpha y_\alpha \tag{3.5}$$

Recall that the Finsler metric is defined as $f_{\mu\nu} = \frac{1}{2}(\partial^2 F^2/\partial y^\mu\, \partial y^\nu)$. This implies immediately that

$$f_{\mu\nu} = \partial y_\mu/\partial y^\nu \tag{3.6}$$

and also, $y_\mu = f_{\mu\nu} y^\nu$.

The partial derivative of (3.5) with respect to $\bar{y}^\nu$ produces

$$\frac{\partial\bar{y}_\mu}{\partial\bar{y}^\nu} = \bar{f}_{\mu\nu} = Y_\mu^\alpha \frac{\partial y_\alpha}{\partial\bar{y}^\nu} + \frac{\partial Y_\mu^\alpha}{\partial\bar{y}^\nu} y_\alpha$$

The application of (3.2) and (3.6) gives

$$\bar{f}_{\mu\nu} = Y_\mu^\alpha Y_\nu^\beta f_{\alpha\beta} + Y_\nu^\beta \frac{\partial Y_\mu^\alpha}{\partial y^\beta} y_\alpha$$

So the Finsler metric is not a tensor under (3.1) unless the condition

$$\frac{\partial Y_\nu^\alpha}{\partial y^\beta} y_\alpha = 0 \tag{3.7}$$

holds. This is called the "metric condition" by Asanov (1985, p. 42).

For a vector which is contravariant under the y transformation, $\bar{q}^\mu = Y^{*\mu}_\nu q^\nu$, a covariant derivative is

$$D_\alpha q^\mu = \frac{\partial q^\mu}{\partial x^\alpha} + L^\mu_{\alpha\nu} q^\nu \tag{3.8}$$

which satisfies $\bar{D}_\alpha \bar{q}^\mu = Y^{*\mu}_\nu D_\alpha Q^\nu$.

The new connection L must transform as

$$\bar{L}^\mu_{\alpha\beta} = Y^\delta_\beta L^\nu_{\alpha\delta} Y^{*\mu}_\nu - Y^\nu_\beta \frac{\partial Y^{*\mu}_\nu}{\partial x^\alpha} \tag{3.9}$$

A significant difference between this and the x transformation is that $L^\mu_{\alpha\beta}$ is explicitly not symmetric in the lower two indices.

The connection L is similar to the connection K of Ikeda (1985). A connection which would correspond to the L of Ikeda's notation is not applicable here because of the metric condition.

The derivative of the Finsler metric tensor which is covariant under this transformation is

$$D_\alpha f_{\mu\nu} = \frac{\partial f_{\mu\nu}}{\partial x^\alpha} - L^\beta_{\alpha\nu} f_{\mu\beta} - L^\beta_{\alpha\mu} f_{\beta\nu}$$

A condition $D_\alpha f_{\mu\nu} = 0$ is imposed. This limits the type of connection L which will appear in the theory, but includes cases of physical significance. A permutation of indices leads to

$$\gamma_{\alpha\mu\nu} = \frac{1}{2}\left(\frac{\partial f_{\alpha\mu}}{\partial x^\nu} + \frac{\partial f_{\nu\alpha}}{\partial x^\mu} - \frac{\partial f_{\mu\nu}}{\partial x^\alpha}\right)$$
$$= \tfrac{1}{2}(L_{\alpha\nu\mu} + L_{\mu\nu\alpha} + L_{\nu\mu\alpha} + L_{\alpha\mu\nu} - L_{\mu\alpha\nu} - L_{\nu\alpha\mu}) \tag{3.10}$$

It is evident that the geodesic equation can thus be written as

$$\frac{dy^\mu}{ds} + \gamma^\mu_{\alpha\beta} y^\alpha y^\beta = 0 = \frac{dy^\mu}{ds} + L^\mu_{\alpha\beta} y^\alpha y^\beta + f^{\mu\nu}(L_{\alpha\beta\nu} - L_{\alpha\nu\beta}) y^\alpha y^\beta \tag{3.11}$$

Recall, now, that the line element of a Finsler space is defined by $ds = F(x, dx) = [g_{\mu\nu}(x, y)\, dx^\mu\, dx^\nu]^{1/2}$. Under the transformation (3.1), the line element in the new space becomes $d\bar{s} = [\bar{g}_{\mu\nu}(x, \bar{y})\, dx^\mu\, dx^\nu]^{1/2}$.

An osculating Riemann space exists for each of these two Finsler spaces. The osculating line elements are equal to the corresponding Finsler line elements. A quantity which describes the change of scale from one osculating space to the other (which is induced by the gauge transformation) is $b = ds/d\bar{s}$. This factor b is called the scale function.

In the osculating Riemann spaces a velocity $\tilde{y}^\mu$ can be defined which undergoes a scale transformation rather than a contravariant transformation like (3.1):

$$\tilde{y}^\mu = \frac{dx^\mu}{d\bar{s}} = b\frac{dx^\mu}{ds} = by^\mu \tag{3.12}$$

In general, $\tilde{y}^\mu$ is distinct from $\bar{y}^\mu$. However, as will be seen in the next section, the two vectors can be equated under certain conditions.

A Finsler metric function can be formed using $\tilde{y}^\mu$: $\tilde{F}(x, \tilde{y}) = [\bar{g}_{\mu\nu}\tilde{y}^\mu\tilde{y}^\nu]^{1/2}$. Under the scale change,

$$\begin{aligned}\tilde{F}(x, \tilde{y}) &= [\bar{g}_{\mu\nu}\tilde{y}^\mu\tilde{y}^\nu]^{1/2} = [\bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu]^{1/2}/d\bar{s}\\ &= [g_{\mu\nu}\, dx^\mu\, dx^\nu]^{1/2}/ds = [g_{\mu\nu}y^\mu y^\nu]^{1/2} = F(x, y)\end{aligned}$$

So this F is also scalar, which is equivalent to $\tilde{y}_\mu\tilde{y}^\mu = y_\mu y^\mu$.

Variation of $\tilde{F}$ produces a geodesic equation

$$\frac{d\tilde{y}^\mu}{d\bar{s}} + \bar{\gamma}^\mu_{\alpha\beta}\tilde{y}^\alpha\tilde{y}^\beta = 0 \tag{3.13}$$

This is equivalent to equation (8) of Beil (1987).

It should be emphasized that the transformations described in this section are of a fundamentally different type than the usual passive or active coordinate transformations. Under the transformations (3.1), scalar products are preserved even though the measurement scale changes. The scale change is reasonable, however, since it reflects the change from one metric to another. A different metric implies different scales for measurement due to the fact that measurements are done in frames with different velocities. Essentially, all spaces which are related to each other by (3.4) are physically equivalent. The tangent space transformations involve a velocity profile associated with an observer. It will be possible, as will be shown below, to relate a class of these transformations directly to acceleration.

In particular, it is of interest to investigate metrics which are equivalent to locally flat space, that is, which are related by a Y transformation to the Lorentz metric $\eta_{\mu\nu}$. This is the subject of the next section.

4. EQUIVALENCE, GAUGE TRANSFORMATIONS, AND FIELDS

This section considers the special case where the original tensor used to produce the Finsler metric function is locally Lorentzian at the points x under consideration, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} \tag{4.1}$$

This is, of course, an ordinary Riemannian space.

Under tangent space transformations of the type discussed in Section 3, the g tensor becomes

$$\tilde{g}_{\mu\nu}(x, \bar{y}) = Y^{\beta}_{\mu} Y^{\alpha}_{\nu} \eta_{\alpha\beta} \tag{4.2}$$

According to the traditional equivalence principle, for any Riemannian metric there is a coordinate transformation which takes the metric to $\eta_{\mu\nu}$ at a point x. Actually, there is a broader theorem of Fermi, discussed in Levi-Civita (1977, p. 167), which says that a transformation exists which takes the metric to $\eta_{\mu\nu}$ all along a given world line, not just at a single point. So in the context of coordinate transformations in Riemann space any metric can be transformed to a local inertial frame along a world line.

It is conjectured that a similar theorem exists for the tangent transformations of Finsler spaces. That is, for any $\tilde{g}_{\mu\nu}$ there exists the inverse of (4.2) which results in the metric $\eta_{\mu\nu}$ in the neighborhood of a world line. This type of generalized equivalence has been discussed by Mack (1981, p. 142).

Actually, the theorem is not necessary for the subsequent development here. Instead, it is simply stated that the spaces of interest are those which can be constructed from Lorentz metrics by (4.2). It turns out that spaces of physical interest can be generated in this manner. Thus, any space considered which is obtained through (4.2) where the inverse of Y^{α}_{ν} exists satisfies a Fermi-like theorem for tangent space transformations. The transformation to an inertial frame is just the inverse of (4.2).

It should be noted that the world line in the inertial space is not necessarily a geodesic. Thus, a test particle in the Lorentz space with this world line may be following a curved (nongeodesic) path. Ordinarily it is said in this instance that the particle is under the influence of some external force.

The alternative point of view advocated here is that when a particle initially assumed to be in a Lorentz frame is observed to be moving on a curved path, the behavior does not necessarily have to be explained by an external force term added to the equation of motion. The alternative explanation is that the motion can equally well be accounted for by a new metric which would result from a gauge transformation like (4.2). Thus, the general relativistic idea of space-time curvature determining the path of a test particle is broadened to include fields other than gravitation, for example, the electromagnetic field.

A geodesic equation in a new metric provides an equation of motion which can be identical with the equation of motion using external force in the Lorentz space. This will be demonstrated presently.

This treatment has the conceptual advantage that the external fields are not added *ad hoc* to the Lagrangian but are included directly in the metric. The Finsler metric function F itself plays the role of the Lagrangian.

The conditions under which this can occur have been given by Yasuda (1981). Also, extended Lagrangian theories are possible (for example, Miron and Radivoiovici-Tatoiu, 1989) which are generalizations of Finsler theory.

Various consequences of the assumption (4.1) are now investigated.

The first is obtained from an examination of (2.1) in the form

$$\bar{f}_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\partial \bar{g}_{\mu\alpha}}{\partial \bar{y}^{\nu}} \bar{y}^{\alpha} + \frac{\partial \bar{g}_{\alpha\nu}}{\partial \bar{y}^{\mu}} \bar{y}^{\alpha} + \frac{1}{2} \frac{\partial^2 \bar{g}_{\alpha\beta}}{\partial \bar{y}^{\mu} \partial \bar{y}^{\nu}} \bar{y}^{\alpha} \bar{y}^{\beta}$$

The second term, considering (4.2), is

$$\frac{\partial \bar{g}_{\mu\alpha}}{\partial \bar{y}^{\nu}} \bar{y}^{\alpha} = \eta_{\delta\gamma} Y^{\gamma}_{\mu} \frac{\partial Y^{\delta}_{\alpha}}{\partial \bar{y}^{\nu}} \bar{y}^{\alpha} + \eta_{\delta\gamma} \frac{\partial Y^{\gamma}_{\mu}}{\partial \bar{y}^{\nu}} Y^{\delta}_{\alpha} \bar{y}^{\alpha}$$

But the metric condition (3.7) can be written as $(\partial Y^{\mu}_{\alpha}/\partial y^{\beta}) y^{\alpha} = 0$, which is equivalent to $(\partial Y^{\mu}_{\alpha}/\partial \bar{y}^{\beta}) \bar{y}^{\alpha} = 0$.

These results imply that $(\partial \bar{g}_{\alpha\mu}/\partial \bar{y}^{\nu}) \bar{y}^{\alpha} = 0$ and that $\bar{f}_{\mu\nu} = \bar{g}_{\mu\nu}$.

Transformations where Y is a function of x only and not a function of y obviously satisfy the metric condition. These are sometimes called the K-group or linear transformations. When (4.1) is assumed, these transformations imply Riemannian spaces. The case is still of interest, however, since the theory then describes how these gauge transformations lead from one general relativistic space to another.

For the Lorentz metric $\eta_{\mu\nu}$ the connection L is zero and in the transformed space (3.9) becomes

$$\bar{L}^{\mu}_{\alpha\beta} = -Y^{\nu}_{\beta} \frac{\partial Y^{*\mu}_{\nu}}{\partial x^{\alpha}}$$

For spaces satisfying the metric condition the Christoffel connection is

$$\bar{\gamma}_{\nu\alpha\beta} = \frac{1}{2} \eta_{\delta\gamma} \left(\frac{\partial Y^{\delta}_{\nu}}{\partial x^{\alpha}} Y^{\gamma}_{\beta} + Y^{\delta}_{\nu} \frac{\partial Y^{\gamma}_{\beta}}{\partial x^{\alpha}} + \frac{\partial Y^{\delta}_{\alpha}}{\partial x^{\beta}} Y^{\gamma}_{\nu} + Y^{\delta}_{\alpha} \frac{\partial Y^{\gamma}_{\nu}}{\partial x^{\beta}} - \frac{\partial Y^{\delta}_{\alpha}}{\partial x^{\nu}} Y^{\gamma}_{\beta} - Y^{\delta}_{\alpha} \frac{\partial Y^{\gamma}_{\beta}}{\partial x^{\nu}} \right) \tag{4.3}$$

since $\bar{f}_{\mu\nu} = Y^{\delta}_{\mu} Y^{\gamma}_{\nu} \eta_{\delta\gamma}$.

It is apparent that $Y^{*\mu}_{\gamma} \bar{f}_{\mu\nu} = Y^{\delta}_{\nu} \eta_{\gamma\delta}$.

Also, from (3.3), $(\partial Y^{\nu}_{\beta}/\partial x^{\alpha}) Y^{*\mu}_{\nu} = -Y^{\nu}_{\beta} (\partial Y^{*\mu}_{\nu}/\partial x^{\alpha})$.

These results are combined to show that

$$\bar{L}_{\nu\alpha\beta} = \bar{L}^{\mu}_{\alpha\beta} \bar{f}_{\mu\nu} = \frac{\partial Y^{\gamma}_{\beta}}{\partial x^{\alpha}} Y^{*\mu}_{\gamma} \bar{f}_{\mu\nu} = \frac{\partial Y^{\gamma}_{\beta}}{\partial x^{\alpha}} Y^{\delta}_{\nu} \eta_{\gamma\delta} \tag{4.4}$$

which implies, using (4.3), that

$$\bar{\gamma}_{\nu\alpha\beta} = \tfrac{1}{2} (\bar{L}_{\beta\alpha\nu} + \bar{L}_{\nu\alpha\beta} + \bar{L}_{\nu\beta\alpha} + \bar{L}_{\alpha\beta\nu} - \bar{L}_{\beta\nu\alpha} - \bar{L}_{\alpha\nu\beta}) \tag{4.5}$$

This checks with (3.10).

Another useful result is

$$\frac{\partial \bar{f}_{\mu\nu}}{\partial x^{\alpha}} = \bar{L}_{\nu\alpha\mu} + \bar{L}_{\mu\alpha\nu} = \bar{\gamma}_{\nu\alpha\mu} + \bar{\gamma}_{\mu\alpha\nu}$$

This is the same as $\bar{D}_{\alpha}\bar{f}_{\mu\nu} = 0$.

The geodesic equation, considering (3.11), is

$$\frac{d\bar{y}^{\mu}}{d\bar{s}} + \bar{\gamma}^{\mu}_{\alpha\beta}\bar{y}^{\alpha}\bar{y}^{\beta} = 0 = \frac{d\bar{y}^{\mu}}{d\bar{s}} + \bar{L}^{\mu}_{\alpha\beta}\bar{y}^{\alpha}\bar{y}^{\beta} + \bar{f}^{\mu\nu}(\bar{L}_{\beta\alpha\nu} - \bar{L}_{\beta\nu\alpha})\bar{y}^{\alpha}\bar{y}^{\beta} \tag{4.6}$$

Now, for the present case (3.8) becomes

$$\bar{D}_{\alpha}\bar{q}^{\mu} = \frac{\partial \bar{q}^{\mu}}{\partial x^{\alpha}} + \bar{L}^{\mu}_{\alpha\beta}\bar{q}^{\beta}$$

When this is contracted with $\bar{y}^{\alpha}$ and then $\bar{q}^{\mu}$ is replaced by $\bar{y}^{\mu}$,

$$\bar{D}_{\alpha}\bar{y}^{\mu}\bar{y}^{\alpha} = \frac{d\bar{y}^{\mu}}{d\bar{s}} + \bar{L}^{\mu}_{\alpha\beta}\bar{y}^{\beta}\bar{y}^{\alpha}$$

This can be substituted in (4.6) to produce a version of the geodesic equation,

$$\bar{D}_{\alpha}\bar{y}^{\mu}\bar{y}^{\alpha} + \bar{f}^{\mu\nu}(\bar{L}_{\beta\alpha\nu} - \bar{L}_{\beta\nu\alpha})\bar{y}^{\alpha}\bar{y}^{\beta} = 0 \tag{4.7}$$

Another form of the geodesic equation can be obtained by a particular assumption concerning the two vectors $\bar{y}^{\mu}$ and $\tilde{y}^{\mu}$. This is simply that, since both vectors represent a velocity in the transformed Riemannian space, they can be equal. That is,

$$\bar{y}^{\mu} = Y^{*\mu}_{\nu}y^{\nu} = by^{\mu} = \tilde{y}^{\mu} \tag{4.8}$$

This has the form of an eigenvalue equation for the transformation matrix. It limits Y^{μ}_{ν} to certain types, but, as will be seen, these types are of physical interest.

Note that (4.8) can be introduced only in the context of the osculating Riemann spaces. It would not generally be consistent with the homogeneity requirements of the Finsler spaces where y^{μ} is an independent variable.

The effect of (4.8) is expressed compactly by returning to (4.6):

$$\begin{gathered} \frac{d\bar{y}^{\mu}}{ds} + \bar{L}^{\mu}_{\alpha\beta}y^{\alpha}y^{\beta} + \bar{f}^{\mu\nu}(\bar{L}_{\beta\alpha\nu} - \bar{L}_{\beta\nu\alpha})y^{\alpha}y^{\beta} = 0 \\ \frac{d\bar{y}^{\mu}}{ds} = \frac{dy^{\mu}}{ds} + \frac{1}{b}\frac{db}{ds}y^{\mu} \end{gathered} \tag{4.9}$$

The quantities $\bar{L}$ and $\bar{f}$ can be expressed directly in terms of the transformation matrices as given above. So (4.9) is an equation of motion for a particle in the inertial space which is written in terms of a tangent space transformation. The $\bar{L}$ terms represent a force on the particle which is imposed by the new metric $\bar{f}_{\mu\nu}$.

At this juncture a number of mathematical results could be listed related to the nonlinear connection $\bar{N}^{\mu}_{\nu}$ and the vertical connection $\bar{C}^{\nu}_{\alpha\beta}$ as well as several curvatures. These developments are postponed to future work.

5. GAUGE TRANSFORMATION EXAMPLES

In this section particular examples of gauge transformations of the type (4.2) are presented. These will illustrate how the above theory might be applied. The first two examples are linear or K-group transformations where Y is dependent on x only.

The first one is

$$Y^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} - B^{-2}[1-(1+kB^2)^{1/2}]B^{\alpha}B_{\mu} \tag{5.1}$$

The vector $B^{\alpha} = B^{\alpha}(x)$ is defined in the original space with metric $\eta_{\alpha\beta}$ so that $B^2 = \eta_{\alpha\beta}B^{\alpha}B^{\beta} = B_{\alpha}B^{\alpha}$.

When (5.1) is used in (4.2), we obtain

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + kB_{\mu}B_{\nu}$$

$$\bar{F}^2(x, \bar{y}) = (\eta_{\mu\nu} + kB_{\mu}B_{\gamma})\bar{y}^{\mu}\bar{y}^{\nu}$$

This general class of Finsler metrics for $B^{\alpha} = B^{\alpha}(x, y)$ is described in Beil (1989). There the geodesic equation for one of these metrics is shown to imply the Lorentz equation for a charged particle in an electromagnetic field.

For the case $B^{\alpha} = B^{\alpha}(x)$ the transformation is linear and the resulting space is Riemannian. This space is discussed in Beil (1987). The metric function can be formed as in Section 3 using the velocity vector $\tilde{y}^{\mu}$. The Finsler metric is just $\bar{f}_{\mu\nu} = \eta_{\mu\nu} + kB_{\mu}B_{\nu}$ with a contravariant form $\bar{f}^{\mu\nu} = \eta^{\mu\nu} - k(1+kB^2)^{-1}B^{\mu}B^{\nu}$.

A derivation is given in Beil (1987) which shows that the geodesic equation is identical with the Lorentz charged particle equation under the conditions

$$B_{\alpha}v^{\alpha} = e/mck \tag{5.2}$$

$$B_{\mu} = A_{\mu} + \partial\Lambda/\partial x^{\mu} \tag{5.3}$$

The vector A_{μ} is the electromagnetic potential of the field external to the particle and $v^{\mu} = dx^{\mu}/d\tau = cy^{\mu}$ is the particle velocity.

An alternate derivation using the ideas of the present work can be obtained by starting with (4.8). For the transformation (5.1) the condition (4.8) becomes

$$\{\delta^\mu_\nu - B^{-2}[1-(1+kB^2)^{-1/2}]B^\mu B_\nu\}v^\nu = [1+kc^{-2}(B_\alpha v^\alpha)^2]^{-1/2}v^\mu$$

with $v_\alpha v^\alpha = c^2$. The eigenvalue condition is satisfied if $B_\mu = \beta v_\mu$, where β is some constant. This implies $B_\alpha v^\alpha = \beta c^2$ and $B_\alpha B^\alpha = \beta^2 c^2$.

The conditions (5.3) and $B_\mu = \beta v_\mu$ imply that for any external potential A_μ there is an electromagnetic gauge transformation which results in a potential vector B_μ which is parallel to the velocity vector of the particle.

The condition that B_μ be parallel to v_μ is a boundary condition which is valid only in the context of the Riemann space osculating to the Finsler space. It does not imply that B_μ is velocity dependent and is introduced only after the computation of the Finsler metric and the variational process which produces the geodesic equation.

There is a remarkable correspondence here with the electron theory of Dirac (1952). Dirac proposed a gauge condition on the potential A_μ which would set it to be proportional to v_μ plus another term which he related to the vorticity of an electron stream.

Actually, the idea of the potential being equal to a velocity term plus a second gaugelike term is in line with standard electromagnetic theory. The equation

$$mv_\mu = \frac{\partial S}{\partial x^\mu} - \frac{e}{c}A_\mu \tag{5.4}$$

is the usual relation between the potential and the action S.

There have been recent geometric discussions of the relation of the electromagnetic potential to velocity by Parrott (1987) and Schweizer (1990).

The geodesic equation in the inertial space is computed for (5.1) starting with

$$\bar{L}_{\nu\alpha\mu} = B^{-2}[1-(1+kB^2)^{1/2}]\left(\frac{\partial B_\mu}{\partial x^\alpha}B_\nu - B_\mu\frac{\partial B_\nu}{\partial x^\alpha}\right) + k\frac{\partial B_\mu}{\partial x^\alpha}B_\nu$$

$$\bar{f}^{\mu\nu} = \eta^{\mu\nu} - k(1+kB^2)^{-1}B^\mu B^\nu$$

When these expressions are substituted into (4.9) the result is

$$\frac{dv^\mu}{d\tau} + k[\eta^{\mu\nu} - k(1+kB^2)^{-1}B^\mu B^\nu]\left[B_\nu\frac{\partial B_\alpha}{\partial x^\beta} + \left(\frac{\partial B_\nu}{\partial x^\alpha} - \frac{\partial B_\alpha}{\partial x^\nu}\right)B_\beta\right]v^\alpha v^\beta = 0$$

If $B^\mu = \beta v^\mu$ as above, then

$$\frac{\partial B_\alpha}{\partial x^\beta}v^\alpha v^\beta = \beta\frac{dv_\alpha}{d\tau}v^\alpha = 0$$

Also, the term involving B^ν vanishes since a symmetric part $v^\nu v^\alpha$ is multiplied by the part which is antisymmetric in the same indices. This produces

$$\frac{dv^\mu}{d\tau}+k\beta c^2\eta^{\mu\nu}\left(\frac{\partial B_\nu}{\partial x^\alpha}-\frac{\partial B_\alpha}{\partial x^\nu}\right)v^\alpha=0$$

The use of (5.2) and (5.3) then gives the Lorentz equation:

$$\frac{dv^\mu}{d\tau}+\frac{e}{mc}\eta^{\mu\nu}F_{\alpha\nu}v^\alpha=0$$

The transformation (5.1) can also be modified slightly to produce a negative sign in the metric, $\bar{g}_{\mu\nu}=\eta_{\mu\nu}-kB_\mu B_\nu$.

The equation of motion is

$$\frac{dv^\mu}{d\tau}-kB_\gamma v^\gamma\eta^{\mu\nu}\left(\frac{\partial B_\nu}{\partial x^\alpha}-\frac{\partial B_\alpha}{\partial x^\nu}\right)v^\alpha=0 \tag{5.5}$$

This allows the use of conditions a bit different from (5.2) and (5.3):

$$B_\alpha v^\alpha=-c/k^{1/2} \qquad B_\mu=\frac{e}{k^{1/2}c^2m}\left(A_\mu-\frac{c}{e}\frac{\partial S}{\partial x^\mu}\right) \tag{5.6}$$

These conditions lead directly to the Lorentz equation. This is a "natural" choice of the electromagnetic gauge in a sense which is discussed in Beil (1991).

Equations (5.6) imply that the g tensor could be written as $\bar{g}_{\mu\nu}=\eta_{\mu\nu}-c^{-2}v_\mu v_\nu$. This type of metric goes back to discussions by Synge (1971) and has recently been studied by Kawaguchi and Miron (1989) in the context of generalized Lagrange spaces. The spaces produced by this metric are of a class which is broader than traditional Finsler spaces.

There is also a correspondence between the theory of metrics produced by (5.1) and Kaluza-Klein theory. A comparison between this type of metric and Kaluza theory is given in Beil (1987). Basically, in this theory the electromagnetic field corresponds to a connection instead of a curvature as in Kaluza theories.

The second example of a K-type gauge transformation is

$$Y^\alpha_\mu=\lambda(x)\delta^\alpha_\mu \tag{5.7}$$

which leads to $\bar{g}_{\mu\nu}=\lambda^2\eta_{\mu\nu}$. This class of spaces is discussed by Tavakol and Van den Bergh (1986) and also in Nishioka (1984). These look like the familiar Weyl spaces (for example, Adler *et al.*, 1975, p. 491). There is also a correspondence with the theory of conformal transformations (Fulton *et al.*, 1962).

There is a significant difference, however, between the present theory and the Weyl theory as it is usually developed. Here, the metric tensor transformation is $\bar{f}_{\mu\nu} = \lambda^2 \eta_{\mu\nu}$ and a general contravariant vector transforms as

$$\bar{q}^\mu = Y^{*\mu}_\nu q^\nu = \lambda^{-1} \delta^\mu_\nu q^\nu = \lambda^{-1} q^\mu$$

The length of this vector is $\bar{f}_{\mu\nu} \bar{q}^\mu \bar{q}^\nu = \eta_{\mu\nu} q^\mu q^\nu$. So the lengths of vectors are unchanged under these gauge transformations.

In the Weyl theory the transformation is applied only to the metric tensor, so that the vector length is changed by the scale factor λ^2. This, historically, was a major objection to the Weyl theory (Adler *et al.*, 1975).

Since the lengths of vectors are invariant here, one has a Weyl-type theory without the above drawback. This point is brought out in Nishioka (1984).

The line element ds is still subject to a scale change since it is the length of the increment dx^μ. Thus, $d\bar{s} = \lambda \, ds$. This corresponds to the fact that a field is "turned on" by the gauge transformation or, alternatively, that a reference frame with a different velocity is used. This is in contrast to the conformal or Weyl theories, where the transformation is a coordinate transformation and the scale change is induced by the coordinate change coupled with the transport of a vector from one point to another. Though some of the equations are similar, there is a fundamental difference between the two theories. See Fulton *et al.* (1962) for a thorough discussion of conformal theories.

For transformations of the type (5.7) the velocity vector transforms contravariantly as well as by the scale change:

$$\bar{v}^\mu = Y^{*\mu}_\nu v^\nu = \lambda^{-1} v^\mu$$

Because $\lambda = b^{-1}$, the eigenvalue equation (4.8) holds automatically.

The connections are easily obtained:

$$\bar{L}_{\nu\alpha\beta} = \lambda \frac{\partial \lambda}{\partial x^\alpha} \eta_{\beta\nu}$$

$$\bar{\gamma}_{\nu\alpha\beta} = \lambda \left(\frac{\partial \lambda}{\partial x^\alpha} \eta_{\nu\beta} + \frac{\partial \lambda}{\partial x^\beta} \eta_{\alpha\nu} - \frac{\partial \lambda}{\partial x^\nu} \eta_{\alpha\beta} \right)$$

The geodesic equation is derived from (4.9):

$$\frac{dv^\mu}{d\tau} + \frac{1}{\lambda} \frac{d\lambda}{d\tau} v^\mu - c^2 \eta^{\mu\nu} \frac{1}{\lambda} \frac{\partial \lambda}{\partial x^\nu} = 0$$

Now, a possible choice for λ is

$$\lambda(x) = 1 + 2q_\nu x^\nu + q^2 x^2$$

where q is a constant vector. This form is suggested by the traditional expression for the conformal transformation. One has, then,

$$\frac{1}{\lambda}\frac{\partial\lambda}{\partial x^\alpha}=\frac{2}{\lambda}(q_\alpha+q^2x_\alpha)=\frac{a_\alpha}{c^2}$$

which defines a vector a_α with $a^2=4c^2q^2/\lambda$.

The geodesic equation in terms of a_α would be

$$\frac{dv^\mu}{d\tau}+\frac{v_\alpha a^\alpha}{c^2}v^\mu-a^\mu=0$$

But if $v_\alpha a^\alpha=0$, then $a^\mu=dv^\mu/d\tau$.

So the result is that this type of gauge transformation leads to an expression for a transformation to a frame with an acceleration a^μ.

Finally, a general class of transformations which are not of the K type, that is, where Y^* is dependent on y^μ, is mentioned. These satisfy the requirements of homogeneity as well as the condition $\partial Y_\nu^{*\mu}/\partial y^\alpha=\partial Y_\alpha^{*\mu}/\partial y^\nu$ and the metric condition.

The transformations are

$$Y_\nu^{*\mu}=g_1(x)\delta_\nu^\mu+g_2(r)A^\mu A_\nu+g_3(r)(A^\mu B_\nu+A_\nu B^\mu)+g_4(r)B^\mu B_\nu$$

The vectors A^μ and B^μ depend only on x. The y dependence enters in the variable $r=(A_\alpha y^\alpha)/(B_\beta y^\beta)$, which is clearly of zero homogeneity in y^μ. The general functions of r, g_2, g_3, and g_4 are required to satisfy the conditions

$$g_3'=-rg_2', \qquad g_4'=-rg_3'$$

where the prime denotes a derivative with respect to the argument.

These transformations appear to lead to Finsler metrics which have not previously been investigated.

A number of other gauge transformation types in addition to the ones given above are possible. The theory provides a systematic method for generating metrics by choosing various forms of Y. There may be several applications in various physical contexts.

6. DISCUSSION

Two principal results should be emphasized:

First, it has been demonstrated how a unified theory of gravitation and electromagnetism is implied by a gauge transformation in Finsler tangent space. The electromagnetic potential is directly related to the parameters of the transformation. The transformation produces a new metric which has an equation of motion that is the Lorentz equation of charged particles.

This means that there is a generalized equivalence (through the gauge transformations) of inertial spaces to spaces which contain electromagnetic fields. The electromagnetic fields arise from connections of the transformed metric rather than from curvatures, as in other unified theories.

Second, it has been shown how the old Weyl or conformal theories can be rehabilitated to overcome the objection of noninvariance of the lengths of vectors. One has all the beauty of the original Weyl theory without nonphysical effects. A particular example was given in which the acceleration of a frame is directly related to a gauge transformation.

ACKNOWLEDGMENTS

I would like to thank G. S. Asanov, R. Miron, and T. Kawaguchi, who read and commented on drafts of this paper. In particular, Prof. Miron made significant contributions to the presentation in Section 2 and Dr. Asanov's advice was extremely helpful for revisions to Section 3. I am grateful to both Dr. Asanov and Prof. Miron for continuing encouragement in this work.

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